Chen simple modules and Prüfer modules over Leavitt path algebras

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Overview

- Brief review of Leavitt path algebras
- Chen simple modules
- $\mathrm{Ext}^1_{L_K(E)}(S, T)$ for varous simple $L_K(E)$-modules S, T
- Prüfer modules
- Injective modules over $L_K(E)$

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- $e^*e' = \delta_{e,e'}r(e)$ for any $e, e' \in E^1$
- $v = \sum_{\{e \in E^1 | s(e)=v\}} ee^*$ (for any $v \in E^0$ with $0 < |s^{-1}(v)| < \infty$.)

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- Let M be a left $L_K(E)$-module and $m \in M$. Denote by

$$\hat{\rho}_m : L_K(E) \to M, \quad r \mapsto rm.$$

For a vertex $v \in E^0$, denote by

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Infinite paths

- An *infinite path in E* is a sequence $p = e_1 e_2 e_3 \cdots$, where $e_i \in E^1$ for all $i \in \mathbb{N}$, and for which $s(e_{i+1}) = r(e_i)$ for all $i \in \mathbb{N}$.

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- If $p, q \in E^\infty$, p and q are *tail equivalent* ($p \sim q$) if there exist integers m, n for which $\tau_{>m}(p) = \tau_{>n}(q)$
- $p \in E^\infty$ is *rational* if $p \sim c^\infty$ for some closed path c. $p \in E^\infty$ is *irrational* if it is not rational.

Example

Let R_2 denote the graph

$$\begin{tikzcd} \bullet^{v} \arrow[loop left, "e"] \arrow[loop right, "f"] \end{tikzcd} .$$

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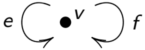

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- $q = efeffefffeffffe \cdots$ is an irrational infinite path in R_2^∞.

Chen simple modules

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Let $p \in E^\infty$. Let $V_{[p]}$ denote the K-vector space with basis the distinct elements of E^∞ which are tail-equivalent to p. For any $v \in E^0$, $e \in E^1$, and $q = f_1 f_2 f_3 \cdots$ with $q \sim p$, define

$$v \cdot q = \begin{cases} q & \text{if } v = s(f_1) \\ 0 & \text{otherwise} \end{cases} \quad e \cdot q = \begin{cases} eq & \text{if } r(e) = s(f_1) \\ 0 & \text{otherwise,} \end{cases} \quad e^* \cdot q = \begin{cases} \tau_{>1}(q) & \text{if } e = f_1 \\ 0 & \text{otherwise} \end{cases}$$

The K-linear extension of this action endows $V_{[p]}$ with the structure of a left $L_K(E)$-module.

Chen simple modules

Theorem: Let $p \in E^{\infty}$. Them the left $L_K(E)$-module $V_{[p]}$ is simple. If $p, q \in E^{\infty}$, then $V_{[p]} \cong V_{[q]}$ as left $L_K(E)$-modules if and only if $p \sim q$, if and only if $V_{[p]} = V_{[q]}$.

Idea: A linear combination of distinct paths tail equivalent to p can be reduced to a single nonzero term by appropriate multiplication. Then any path tail equivalent to p can be generated from this single term via the module action.

X.W. Chen, "Irreducible representations of Leavitt path algebras", Forum Math. **27**(1), 2015, 549–574.

Chen simple modules

Note: Let $w \in E^0$ be a sink. We consider $w = w^\infty$ as an element in E^∞. The Chen simple module $V_{[w^\infty]}$ coincides with the ideal $L_K(E)w$.

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- For $q = efeffefffeffffe\cdots$, $V_{[q]}$ is a Chen simple module generated by an irrational infinite path.

Projective resolutions of Chen simple modules

Reminder: For a left R-module M, a *projective resolution* of M is an exact sequence

$$\cdots P_n \to P_{n-1} \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

where each P_i is a projective left R-module.

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Remark: Type (1) is trivial, since w is an idempotent and so the left ideal $L_K(E)w$ is a projective left $L_K(E)$-module. Type (3) is interesting, but we won't need it in the rest of the lecture, so discussion omitted.

Type (2)

Theorem: Let c be a basic closed path in E, with $v = s(c)$.

1. A projective resolution of $V_{[c^\infty]}$ is given by

$$0 \longrightarrow L_K(E)v \xrightarrow{\rho_{c-v}} L_K(E)v \xrightarrow{\rho_{c^\infty}} V_{[c^\infty]} \longrightarrow 0$$

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In particular, the Chen simple module $V_{[c^\infty]}$ is finitely presented.

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and the Chen simple module $V_{[e^\infty]}$. Then

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are projective resolutions of the finitely presented module $V_{[e^\infty]}$.

Proof

Main points of the proof:

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- The inclusion $\mathrm{Ker}(\rho_{c^\infty}) \subseteq L_K(E)(c - v)$ follows analyzing the shape of the standard form monomials in $\mathrm{Ker}(\rho_{c^\infty})$
- By a degree argument, we get $r(c - v) = 0$ if and only if $r = 0$. So the map $\rho_{c-v} : L_K(E)v \to L_K(E)v$ is a monomorphism of left $L_K(E)$-modules.

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- $\mathrm{Ext}^1_{L_K(E)}(S, T) \neq 0$ if and only if there exists a non-splitting short exact sequence $0 \to T \to N \to S \to 0$
- If w is a sink, then $\mathrm{Ext}^1_{L_K(E)}(V_{[w^\infty]}, M) = 0$ for any M.

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Theorem: (A-, Mantese, Tonolo, 2015) Let E be a finite graph. Let d be a basic closed path in E and let T be a Chen simple module. Then the following are equivalent:

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Corollary: Let E be a finite graph. Let d be a basic closed path. Then $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, V_{[d^\infty]}) \neq 0$. In particular, $V_{[d^\infty]}$ is neither projective, nor injective.

Example

Consider the graph R_2:

Let $q \in R_2^\infty$ and let $T = V_{[q]}$. Let d be a basic closed path in R_2. Since $v = s(d) \in U(T) = \{v\}$, the previous theorem applies and hence $\mathrm{Ext}^1_{L_K(R_2)}(V_{[d^\infty]}, T) \neq 0$.

Proof: main points

Let E be a finite graph. Let d be a basic closed path in E and let T be a Chen simple module. Consider the projective resolution

$0 \longrightarrow L_K(E) \xrightarrow{\hat{\rho}_{d-1}} L_K(E) \xrightarrow{\hat{\rho}_{d^\infty}} V_{[d^\infty]} \longrightarrow 0$ and the resulting standard long exact sequence

$$\mathrm{Hom}_{L_K(E)}(V_{[d^\infty]}, T) \xrightarrow{\hat{\rho}_{d^\infty_*}} \mathrm{Hom}_{L_K(E)}(L_K(E), T) \xrightarrow{\hat{\rho}_{(d-1)_*}} \mathrm{Hom}_{L_K(E)}(L_K(E), T)$$

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$$\xrightarrow{\pi} \mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, T) \longrightarrow \mathrm{Ext}^1_{L_K(E)}(L_K(E), T) \ (=0) \longrightarrow \cdots$$

Proof: main points

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So we get:

Proposition: $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, T) = 0$ if and only if $(d-1)X = t$ has a solution in T for every $t \in T$.

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But then it's not hard to show:

Lemma:

1) Let $T = V_{[q]}$, with $V_{[q]} \neq V_{[d^\infty]}$. Suppose $s(d) \in U(T)$. Let $t \in T$ be "not divisible" by d. Then the equation $(d-1)X = t$ has no solution in T

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In particular, we have

Corollary: For d a cycle in E, the left $L_K(E)$-module $V_{[d^\infty]}$ is (simple and) not injective.

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Prüfer modules

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Corollary: For d a cycle in E, the left $L_K(E)$-module $V_{[d^\infty]}$ is (simple and) not injective.

Question: What is the injective hull of $V_{[d^\infty]}$?

Recall: $\hat{\rho}_{d-1} : L_K(E) \to L_K(E)$ is a monomorphism. (In other words, $d-1$ is not a right zero-divisor in $L_K(E)$.) Moreover,

$$V_{[d^\infty]} \cong L_K(E)/L_K(E)(d-1).$$

Prüfer modules

We look at the standard Prüfer abelian groups for guidance.

p denotes a prime in $\mathbb{Z}$.

$$\mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathbb{Z}/p^2\mathbb{Z} \hookrightarrow \mathbb{Z}/p^3\mathbb{Z} \hookrightarrow \cdots$$

The embedding is $\quad a + p^i\mathbb{Z} \ \mapsto \ pa + p^{i+1}\mathbb{Z}$
The Prüfer p-group is

$$\mathbb{Z}(p^\infty) = \bigcup_{i=1}^{\infty} \mathbb{Z}/p^i\mathbb{Z}$$

Another point of view: $\mathbb{Z}(p^\infty) = \{\frac{a}{p^i} | i \in \mathbb{N}\}$, with addition mod $\mathbb{Z}$.

Prüfer modules

Well-known properties of $\mathbb{Z}(p^\infty)$:

1) $\mathbb{Z}(p^\infty)$ is divisible as a $\mathbb{Z}$-module: for every $z \in \mathbb{Z}$ and $t \in \mathbb{Z}(p^\infty)$ the equation $zX = t$ has a solution in $\mathbb{Z}(p^\infty)$. In particular, $\mathbb{Z}(p^\infty)$ is injective as a $\mathbb{Z}$-module.

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2) The only proper subgroups of $\mathbb{Z}(p^\infty)$ are the $\mathbb{Z}/p^i\mathbb{Z}$ ($i \in \mathbb{N}$). In particular, $\mathbb{Z}(p^\infty)$ has d.c.c., but not a.c.c., on submodules.

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4) $\mathbb{Z}(p^\infty) \ / \ \mathbb{Z}/p^i\mathbb{Z} \ \cong \ \mathbb{Z}(p^\infty)$ for all $i \in \mathbb{N}$.

5) The equation $pX = 1 + p^i\mathbb{Z}$ has no solution in $\mathbb{Z}/p^i\mathbb{Z}$.

Prüfer modules

6) $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}(p^{\infty}))$ is the ring of p-adic integers; think of this as "formal power series in p", with coefficients in $\{0, 1, \ldots, p-1\}$

Prüfer modules

6) $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}(p^\infty))$ is the ring of p-adic integers; think of this as "formal power series in p", with coefficients in $\{0, 1, \ldots, p-1\}$

OR, think of it as an inverse limit of the rings / maps

$$\cdots \to \mathbb{Z}/p^3\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p^1\mathbb{Z}.$$

Prüfer modules

We can do this in general.

Proposition: Suppose $a \in R$ has these two properties:

(1) R/Ra is a simple left R-module, and

(2) for every $i \in \mathbb{N}$, the equation $aX = 1 + Ra^i$ has no solution in R/Ra^i.

Then the direct limit $U_{R,a}$ of the sequence

$$R/Ra \hookrightarrow R/Ra^2 \hookrightarrow R/Ra^3 \hookrightarrow \cdots$$

has structural properties analogous to those for $\mathbb{Z}(p^\infty)$ given above.

Prüfer modules

Now we apply these ideas to the specific case where

$$R = L_K(E), \ a = c - 1$$

where c is a cycle in the finite graph E.

$$L_K(E)/L_K(E)(c-1) \hookrightarrow L_K(E)/L_K(E)(c-1)^2 \hookrightarrow L_K(E)/L_K(E)(c-1)^3 \hookrightarrow \cdots$$

Denote the direct limit of this sequence by $U_{E,c-1}$.

Prüfer modules

We already have property (1):

$L_K(E)/L_K(E)(c-1)$ is a simple left $L_K(E)$-module, because it is isomorphic to $V_{[c^\infty]}$.

For property (2):

Proposition: For any basic closed path c in E, the equation

$$(c-1)X = 1 + L_K(E)(c-1)^n$$

has NO solution in $L_K(E)/L_K(E)(c-1)^n$.

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Idea of proof: Establish a "Division Algorithm by $c-1$" inside $L_K(E)$. (Messy, but relatively straightforward.)

Prüfer modules

Proposition: Let E be a finite graph, let c be a basic closed path in E based at v, and let $U_{E,c-1}$ be the Prüfer module associated to c. Suppose that there exists a cycle $d \neq c$ which connects to v. Then $U_{E,c-1}$ is not injective.

Prüfer modules

Proof: By work on Ext^1 groups described previously (using the hypothesis that d connects to v),

$$\mathrm{Ext}^1(V_{[d^\infty]}, V_{[c^\infty]}) \neq 0.$$

Let α_1 denote $1 + L_K(E)(c-1)$. We get

$$0 \to V_{[c^\infty]} \cong L_K(E)\alpha_1 \hookrightarrow U_{E,c-1} \twoheadrightarrow U_{E,c-1}/L_K(E)\alpha_1 \cong U_{E,c-1} \to 0$$

But $\mathrm{Hom}(V_{[d^\infty]}, U_{E,c-1}) = 0$, because the only simple submodule of $U_{E,c-1}$ is isomorphic to $V_{[c^\infty]} \ncong V_{[d^\infty]}$.

Prüfer modules

This gives the resulting long exact sequence

$$\mathrm{Hom}_{L_K(E)}(V_{[d^\infty]}, V_{[c^\infty]}) \longrightarrow \mathrm{Hom}_{L_K(E)}(V_{[d^\infty]}, U_{E,c-1}) \longrightarrow \mathrm{Hom}_{L_K(E)}(V_{[d^\infty]}, U_{E,c-1}) \ (=0)$$

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Consequently, $\mathrm{Ext}^1(V_{[d^\infty]}, U_{E,c-1}) \neq 0$, so that $U_{E,c-1}$ is not injective.

Prüfer modules

On the other hand what happens when there is NO cycle d which connects to c?

Call such a cycle c *maximal*.

Prüfer modules

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Example: The Toeplitz graph

$$T = \begin{tikzcd} \bullet \arrow[loop left, "c"] \arrow[r] & \bullet \end{tikzcd}$$

(The Leavitt path algebra $L_K(T)$ is isomorphic to the Jacobson algebra $K\langle X, Y | XY = 1\rangle$.)

Prüfer modules

Main Theorem: Let E be a finite graph and let c be a basic closed path in E. Let $U_{E,c-1}$ be the Prüfer module associated to c. Then $U_{E,c-1}$ is injective if and only if c is a maximal cycle.

Prüfer modules

Main Theorem: Let E be a finite graph and let c be a basic closed path in E. Let $U_{E,c-1}$ be the Prüfer module associated to c. Then $U_{E,c-1}$ is injective if and only if c is a maximal cycle.

Moreover, in case $U_{E,c-1}$ is injective, then:

(1) $U_{E,c-1}$ is the injective envelope of the Chen simple module $V_{[c^\infty]}$, and
(2) $\mathrm{End}_{L_K(E)}(U_{E,c-1})$ is isomorphic to the ring $K[[x]]$ of formal power series in x.

Prüfer modules

One direction? Done above.

Other direction?

Prüfer modules

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Other direction?

Two steps: Reduce to the case when c is a source loop. Then prove the result in this case.

Prüfer modules

Proposition:

1) Source elimination is a Morita equivalence, and preserves Prüfer modules.

2) Reduction of a source cycle to a source loop is a Morita equivalence, and preserves Prüfer modules.

Proof: Omitted. Not too difficult.

Prüfer modules

We analyze specific elements.

Proposition: Let c be a source loop. Let $j \in \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$. Then there exists $n \in \mathbb{N}$ such that ${c^*}^n j = 0$.

Proof: It is not hard to show that any nonzero $j \in \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$ is a K-linear combination of elements of the form

$$\alpha\beta^* w\gamma\delta^* \neq 0,$$

where $w \neq s(c)$. Now consider cases.

1) If $\alpha\beta^* w = w$ then $c^*\alpha\beta^* w\gamma\delta^* = c^* w\gamma\delta^* = 0$.

2) If $\alpha\beta^* w = \beta^* w \neq w$ then $s(\beta^*) = r(\beta) \neq s(c)$, otherwise β would be a path which starts in w and ends at $s(c)$, contrary to c being a source loop. Then $c^*\alpha\beta^* w\gamma\delta^* = c^*\beta^* w\gamma\delta^* = 0$.

Prüfer modules

3) In all the other cases $\alpha = c^t \eta_1 \cdots \eta_s$ with $c \neq \eta_1 \in E^1$, $t \geq 0$ and $s \geq 1$. Then

$$(c^{t+1})^* \alpha \beta^* w \gamma \delta^* = (c^{t+1})^* c^t \eta_1 \cdots \eta_s \beta^* w \gamma \delta^* = c^* \eta_1 \cdots \eta_s \beta^* w \gamma \delta^* = 0.$$

Since j is a finite sum of terms of the form $\alpha \beta^* w \gamma \delta^*$, the result follows.

Prüfer modules

Proposition: For any $\ell \in L_K(E) \setminus \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$ and for any $u \in U_{E,c-1}$, there exists $X \in U_{E,c-1}$ such that $\ell X = u$. That is, u is divisible by any element in $L_K(E) \setminus \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$.

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Idea of Proof: It can be shown that

$$\mathrm{Ann}_{L_K(E)}(U_{E,c-1}) = \bigcap_{n \geq 1} L_K(E)(c-1)^n = \langle E^0 \setminus s(c) \rangle.$$

Then using the "Division Algorithm" for $c - 1$ (and some computation) yields the result.

Prüfer modules

Corollary: If $0 \neq u \in U_{E,c-1}$ then $(c^*)^m u \neq 0$ for all $m \in \mathbb{N}$.

Prüfer modules

Corollary: If $0 \neq u \in U_{E,c-1}$ then $(c^*)^m u \neq 0$ for all $m \in \mathbb{N}$.

Proof: Since $c \notin L_K(E)(c-1) \supseteq \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$, by previous Proposition there exists $0 \neq x \in U_{E,c-1}$ with

$$cx = u.$$

We may assume that $s(c)x = x$. Then

$$0 \neq x = s(c)x = c^*cx = c^*u.$$

Repeating the same argument for $0 \neq c^*u \in U_{E,c-1}$, we get $(c^*)^2 u \neq 0$. Now continue.

Prüfer modules

Key Proposition: Let c be a source loop in E. Let I_f be a finitely generated left ideal of $L_K(E)$, and let $\varphi : I_f \to U_{E,c-1}$ be a $L_K(E)$-homomorphism. Then there exists $\psi : L_K(E) \to U_{E,c-1}$ such that $\psi|_{I_f} = \varphi$. Consequently,

$$\mathrm{Ext}^1(L_K(E)/I_f, U_{E,c-1}) = 0.$$

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Proof: **By the result presented in this morning's lecture, we know that $L_K(E)$ is a Bézout ring**, i.e., that every finitely generated left ideal of $L_K(E)$ is principal.

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Proof: **By the result presented in this morning's lecture, we know that $L_K(E)$ is a Bézout ring**, i.e., that every finitely generated left ideal of $L_K(E)$ is principal.

So $I_f = L_K(E)\ell$ for some $\ell \in I_f$.

Prüfer modules

Assume on one hand that $\ell \in \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$, and hence $I_f \leq \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$.

But we know these two things:

1) Any element of $\mathrm{Ann}_{L_K(E)}(U_{E,c-1})$ is annihilated by some ${c^*}^N$, and

2) ${c^*}^n u \neq 0$ for all $0 \neq u \in U_{E,c-1}$ and $n \in \mathbb{N}$.

Prüfer modules

But then for $\varphi \in \mathrm{Hom}_{L_K(E)}(I_f, U_{E,c-1})$ we see that $\varphi(\ell) = 0$. Here's why:

Otherwise, if $\varphi(\ell) \neq 0$, then $(c^*)^n \varphi(\ell) \neq 0$ for all n;

but $\ell \in \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$ gives $(c^*)^N \ell = 0$ for some N, so that $0 = \varphi((c^*)^N \ell) = (c^*)^N \varphi(\ell)$, a contradiction.

And $\varphi(\ell) = 0$ gives $\varphi = 0$, because I_f is generated by ℓ. Thus in this case we must have $\mathrm{Hom}_{L_K(E)}(I_f, U_{E,c-1}) = 0$, and the conclusion follows trivially.

Prüfer modules

Assume on the other hand that $\ell \notin \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$.
But then there exists $x \in U_{E,c-1}$ for which $\ell x = \varphi(\ell)$.

Let $\psi : L_K(E) \to U_{E,c-1}$ be the map ρ_x. Then, for each $i = r\ell \in I_f$, we have

$$\psi(i) = \psi(r\ell) = r\ell\psi(1) = r\ell x = \varphi(\ell) = \varphi(r\ell) = \varphi(i),$$

and so φ extends in this case as well.

Prüfer modules

Proposition: Let E be a finite graph, and c a source loop in E. Then the endomorphism ring of the left $L_K(E)$-module $U_{E,c-1}$ is isomorphic to the ring of formal power series $K[[x]]$.

Proof omitted, but it's not too hard.

Prüfer modules

We need one more tool.

We know the entire lattice of proper submodules of $U_{E,c-1}$ as a left $L_K(E)$-module, it consists precisely of the $L_K(E)/L_K(E)(c-1)^i$.

Prüfer modules

We need one more tool.

We know the entire lattice of proper submodules of $U_{E,c-1}$ as a left $L_K(E)$-module, it consists precisely of the $L_K(E)/L_K(E)(c-1)^i$.

But $U_{E,c-1}$ is a right module over its endomorphism ring S, which is isomorphic to $K[[x]]$.

Proposition: Each $L_K(E)/L_K(E)(c-1)^i$ is a right S-submodule of $U_{E,c-1}$, and these are ALL the right S-submodules of $U_{E,c-1}$. In particular, $(U_{E,c-1})_S$ is artinian.

Proof: Not hard.

Prüfer modules

Here's why we care about the right S-structure of $U_{E,c-1}$:

This property implies that the functor $\mathrm{Ext}^1(-, U_{E,c-1})$ sends direct limits to inverse limits.

(More details: If a module is linearly compact over its endomorphism ring, then it is algebraically compact and hence pure-injective. But for a pure-injective left R-module M, the functor $\mathrm{Ext}^1(-, M)$ sends direct limits to inverse limits.)

Prüfer modules

Finally, we get the result.

Theorem: Let E be a finite graph with source loop c. Then the Prüfer module $U_{E,c-1}$ is injective. Indeed, $U_{E,c-1}$ is the injective envelope of $V_{[c^\infty]}$.

Prüfer modules (Key Prop.) $\mathrm{Ext}^1(L_K(E)/I_f, U_{E,c-1}) = 0.$

Proof: In order to check the injectivity of $U_{E,c-1}$, we apply Baer's Lemma; that is, we need only check that $U_{E,c-1}$ is injective relative to any short exact sequence of the form

$$0 \to I \to L_K(E) \to L_K(E)/I \to 0.$$

This is equivalent to showing that $\mathrm{Ext}^1_{L_K(E)}(L_K(E)/I, U_{E,c-1}) = 0$ for any left ideal I of $L_K(E)$.

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Write $I = \varinjlim I_\lambda$, where the I_λ are the finitely generated submodules of I. It is standard that

$$L_K(E)/I = \varinjlim L_K(E)/I_\lambda.$$

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So now applying the functor $\mathrm{Ext}^1_{L_K(E)}(-, U_{E,c-1})$, we get:

$\mathrm{Ext}^1_{L_K(E)}(L_K(E)/I, U_{E,c-1})$

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$$\mathrm{Ext}^1_{L_K(E)}(L_K(E)/I, U_{E,c-1})$$
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$$\begin{aligned}
&\mathrm{Ext}^1_{L_K(E)}(L_K(E)/I, U_{E,c-1})\\
&\quad = \mathrm{Ext}^1_{L_K(E)}(\varinjlim L_K(E)/I_\lambda, U_{E,c-1})\\
&\quad = \varprojlim \mathrm{Ext}^1(L_K(E)/I_\lambda, U_{E,c-1}) \qquad \text{(by Proposition above)}
\end{aligned}$$

So now applying the functor $\mathrm{Ext}^1_{L_K(E)}(-, U_{E,c-1})$, we get:

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&\mathrm{Ext}^1_{L_K(E)}(L_K(E)/I, U_{E,c-1}) \\
&\quad = \mathrm{Ext}^1_{L_K(E)}(\varinjlim L_K(E)/I_\lambda, U_{E,c-1}) \\
&\quad = \varprojlim \mathrm{Ext}^1(L_K(E)/I_\lambda, U_{E,c-1}) && \text{(by Proposition above)} \\
&\quad = \varprojlim 0 = 0. && \text{(by Key Proposition)}
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\end{aligned}$$

Since $L_K(E)\alpha_1$ is an essential submodule of $U_{E,c-1}$, the last statement follows.